Mister Modulo - Solution

Factorize $a = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$. Then we want k minimal such that

$$10^k \equiv 1 \mod p_1^{ne_1} p_2^{ne_2} \dots p_k^{ne_k}.$$

By the Chinese Remainder Theorem, this is the case if and only if $10^k \equiv 1 \mod p_i^{ne_i}$ for all i. For each such i there is a minimal k_i such that $10^{k_i} \equiv 1 \mod p_i^{ne_i}$, and then the minimal k is the least common multiple of all these k_i. We therefore only have to solve the problem for each prime factor separately. Let p be prime and e a positive integer. We look for the minimal k with

$$10^k \equiv 1 \mod p^{ne}.$$

As $a < 10^3$ and a is coprime with 10, so is p^e. In particular p is odd. There are several approaches to calculate the order of 10 for these specific prime power moduli. We make use of the Lifting-The-Exponent lemma. We want that p^{ne} divides $10^k - 1$. For $n = 1$, p^e is small enough that we can brute-force this order. Call this base order m, so that $p^e \mid 10^m - 1$. As p^e has to divide $10^k - 1$ even for larger n, this implies that $m \mid k$. Write $k = \ell m$. As p is odd, if $\nu_p(m)$ denotes the p-adic order (i.e. how many times p divides m), the Lifting-The-Exponent lemma states that

$$\nu_p(10^{m\ell} - 1) = \nu_p(10^m - 1) + \nu_p(\ell).$$

We want $\nu_p(10^{m\ell} - 1) \geq ne$, and again we can easily compute $\nu_p(10^m - 1)$, say this is o. Then we should choose ℓ such that $\nu_p(\ell) \geq ne - o$. If $ne - o \leq 0$, picking $\ell = 1$ works. Otherwise, the minimal ℓ is $\ell = p^{ne-o}$. Thus the answer is $k = m$ or $k = mp^{ne-o}$. This solves the problem for a single prime power which, as we saw, is enough to solve the general problem as well.